MATH2060 TUT06

6. If $\psi : [a,b] \to \mathbb{R}$ takes on only a finite number of distinct values, is ψ a step function?

Recall (Def 5.4.9)

A function $\varphi : [a,b] \to \mathbb{R}$ is a step function if $\exists$ subintervals I_i, $i = 1, \dots, n$ (not necessarily closed) with

$$\begin{cases} I_i \cap I_j = \phi & \text{for } i \neq j \text{ and} \\ [a,b] = \bigcup_{i=1}^{n} I_i , \end{cases}$$

s.t. $\varphi|_{I_i}$ = constant function on I_i

i.e. $\varphi(x) = k_i \quad \forall x \in I_i$ (for some k_i)

Ans: NO.

Consider $\psi(x) := \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} . \end{cases}$

So ψ can only take values 0 or 1.

However ψ is NOT a step fcn because $\{x : \psi(x) = 1\}$ cannot be expressed as a finite union of subintervals of $[0,1]$.

Suppose ψ is a step fcn.

Then $\exists$ subinterval I of $[0,1]$ s.t. $I \subseteq \{x : \psi(x) = 1\}$

Then, by density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists\, q \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $q \in I$

But $\psi(q) = 0$ contradicting $q \in I \subseteq \{x : \psi(x) = 1\}$ //

7. If $S(f;\dot{\mathcal{P}})$ is any Riemann sum of $f:[a,b]\to\mathbb{R}$, show that there exists a step function $\varphi:[a,b]\to\mathbb{R}$ such that $\int_a^b \varphi = S(f;\dot{\mathcal{P}})$.

Ans: Suppose $\dot{P} = \{[x_{i-1},x_i],t_i\}_{i=1}^n$ is a tagged partition of $[a,b]$.

Then $S(f;\dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$

Define $\varphi:[a,b]\to\mathbb{R}$ by

$$\varphi = \sum_{i=1}^n f(t_i)\cdot\varphi_{[x_{i-1},x_i]}$$

where $\varphi_{[c,d]}(x) := \begin{cases} 1 & \text{if } x\in[c,d] \\ 0 & \text{otherwise.}\end{cases}$ ⇐ elementary step fcn

Then φ is a step fcn and

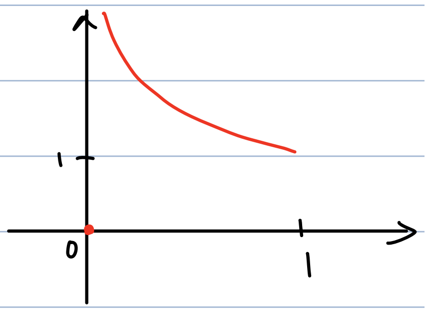
$$\begin{aligned}\int_a^b \varphi &= \sum_{i=1}^n f(t_i)\int_a^b \varphi_{[x_{i-1},x_i]} && \text{(by Thm 7.1.5)}\\ &= \sum_{i=1}^n f(t_i)\cdot(x_i - x_{i-1}) && \text{(by Lemma 7.2.4)}\\ &= S(\varphi;\dot{P})\end{aligned}$$

✓

13. Give an example of a function $f : [a,b] \to \mathbb{R}$ that is in $\mathcal{R}[c,b]$ for every $c \in (a,b)$ but which is not in $\mathcal{R}[a,b]$.

Ans: Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$



Then, $\forall c \in (0,1)$,

f is cts on $[c,1]$

$\Rightarrow f \in \mathcal{R}[c,1]$

However $f \notin \mathcal{R}[0,1]$ since f is not bounded on $[0,1]$ //

17. If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$, show that there exists $c \in [a, b]$ such that $\int_a^b fg = f(c) \int_a^b g$. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.)

Ans: First we show that $\int_a^b g > 0$:

Since g is cts on $[a,b]$, $\exists\, x_* \in [a,b]$ s.t.

$$g(x) \geq g(x_*) =: m > 0 \quad \forall x \in [a,b]$$ EVT

So $\int_a^b g \geq \int_a^b m = m(b-a) > 0$.

Now we prove the Integral MVT:

Since f is cts on $[a,b]$, $\exists\, x_1, x_2 \in [a,b]$

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b].$$ EVT

$$\Rightarrow \quad f(x_1) g(x) \leq f(x) g(x) \leq f(x_2) g(x) \quad \forall x \in [a,b].$$

So, by Thm 7.1.5,

$$f(x_1) \int_a^b g \leq \int_a^b fg \leq f(x_2) \int_a^b g$$

$$\Rightarrow \quad f(x_1) \leq \frac{\int_a^b fg}{\int_a^b g} \leq f(x_2)$$

By Intermediate Value Thm, $\exists\, c \in [a,b]$ s.t. $f(c) = \dfrac{\int_a^b fg}{\int_a^b g}$

i.e. $\int_a^b fg = f(c) \int_a^b g$ //

The conclusion fails if we do not have $g(x) > 0$:

For example, let $f(x) = g(x) = x$ on $[-1,1]$.

Then $\int_{-1}^1 fg = \int_{-1}^1 x^2 = \frac{2}{3} > 0$

but $f(c) \int_{-1}^1 g = f(c) \underbrace{\cancel{\int_{-1}^1 x}}_{0} = 0 \quad \forall c \in [-1,1]$ //

18. Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := (\int_a^b f^n)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.

Ans: If $f \equiv 0$, the result is trivial.

Suppose $f \not\equiv 0$. Since f is cts on $[a,b]$, EVT implies that $\exists\, x_0 \in [a,b]$ s.t. $f(x_0) = \sup\{f(x) : x \in [0,1]\} > 0$

Let $\varepsilon > 0$. s.t. $\varepsilon < f(x_0)$

By continuity, $\exists\, \delta > 0$ s.t

$$|f(x) - f(x_0)| < \varepsilon \quad \forall\, x \in (x_0-\delta, x_0+\delta) \cap [a,b].$$

In particular, $\exists\, [c,d] \subseteq [a,b]$ $(c<d)$ s.t

$$0 < f(x_0) - \varepsilon < f(x) \leq f(x_0) \quad \forall\, x \in [c,d].$$

Now,
$$\int_a^b f^n \leq \int_a^b f(x_0)^n = (b-a) f(x_0)^n$$

and
$$\int_a^b f^n \underset{\substack{\uparrow \\ f \geq 0}}{\geq} \int_c^d f^n \geq \int_c^d \underbrace{(f(x_0) - \varepsilon)}_{>0}{}^n = (d-c)(f(x_0)-\varepsilon)^n$$

Hence, $(d-c)^{1/n}(f(x_0)-\varepsilon) \leq M_n = \left(\int_a^b f^n\right)^{1/n} \leq (b-a)^{1/n} f(x_0) \quad \forall\, n \in \mathbb{N}.$

Note $\lim_{n\to\infty} \alpha^{1/n} = 1 \quad \forall\, \alpha > 0.$

~~Letting $n \to \infty$, $f(x_0) - \varepsilon \leq \lim_{n\to\infty} M_n \leq f(x_0)$~~ $\exists ?$

$$\limsup_{n\to\infty} M_n \leq \limsup_{n\to\infty} (b-a)^{1/n} f(x_0) = f(x_0)$$

$$\Downarrow$$

$$f(x_0) - \varepsilon = \liminf_{n\to\infty} (d-c)^{1/n}(f(x_0)-\varepsilon) \leq \liminf_{n\to\infty} M_n$$

Letting $\varepsilon \to 0^+$, we have $\liminf_{n\to\infty} M_n = \limsup_{n\to\infty} M_n = f(x_0)$

That is $\lim_{n\to\infty} M_n = f(x_0) = \sup\{f(x) : x \in [0,1]\}$

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